LESSON #33 - SEQUENCES
COMMON CORE ALGEBRA II

In Common Core Algebra I, you studied sequences, which are ordered lists of numbers. Sequences are extremely important in mathematics, both theoretical and applied. A sequence is formally defined as a function that has as its domain the set the set of positive integers, i.e. \( \{1, 2, 3, ..., n\} \).

Exercise #1: A sequence is defined by the equation \( a(n) = 2n - 1 \).
(a) Find the first three terms of this sequence, denoted by \( a(1) \), \( a(2) \), and \( a(3) \).
(b) Which term has a value of 53?
(c) Explain why there will not be a term that has a value of 70.

Exercise #2: A sequence is defined explicitly as \( a_n = 3n - 5 \).
(a) What is the value of the 5th term in this sequence?
(b) What is the value of \( a_{12} \)?

(c) With explicit sequence formulas, when you are looking for a specific term in the sequence, what do you need to do?

Recall that sequences can also be described by using recursive definitions. When a sequence is defined recursively, terms are found by operations on previous terms.

Exercise #3: A sequence is defined by the recursive formula: \( f(n) = f(n-1) + 5 \) with \( f(1) = -2 \).
(a) Generate the first five terms of this sequence.
Label each term with proper function notation.
(b) Determine the value of \( f(20) \). Hint – think about how many times you have added 5 to -2.
**Exercise #4:** A sequence is defined recursively as \( a_1 = 2; \ a_n = 3a_{n-1} + 1. \)

(a) What is the value of the second term in the sequence?

(b) What is the value of the fourth term in the sequence?

(c) When you are looking for a specific term in a sequence defined recursively, what must you find first?

**Exercise #5:** For the recursively defined sequence \( t_n = (t_{n-1})^2 + 2n \) and \( t_1 = 2 \), what is the value of \( t_4 \)?

**Exercise #6:** One of the most well-known sequences is the Fibonacci, which is defined recursively using two previous terms. Its definition is given below.

\[
f(n) = f(n-1) + f(n-2) \text{ and } f(1) = 1 \text{ and } f(2) = 1
\]

Generate values for \( f(3), f(4), f(5), \) and \( f(6) \) (in other words, then next four terms of this sequence).

**Exercise #7:** Which of the following would represent the graph of the sequence \( a_n = 2n + 1 \)? Explain your choice.

(1) ![Graph 1](image1)
(2) ![Graph 2](image2)
(3) ![Graph 3](image3)
(4) ![Graph 4](image4)

Explanation:
**Exercise #8:** Match each of the explicit and recursive formulas with its sequence of numbers.

<table>
<thead>
<tr>
<th>Explicit Formula</th>
<th>Recursive Formula</th>
<th>Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) $f(n) = 6(-3)^{n-1}$</td>
<td>A. $f(1) = 6, f(n) = f(n-1) + 6$</td>
<td>5) 6, 9, 12, 15, …</td>
</tr>
<tr>
<td>2) $a_n = 3n + 3$</td>
<td>B. $a_1 = 6, a_n = \frac{1}{2} a_{n-1}$</td>
<td>6) 6, 3, $\frac{3}{2}, \frac{3}{4}$ …</td>
</tr>
<tr>
<td>3) $f(n) = 6n$</td>
<td>C. $f(1) = 6, f(n+1) = -3f(n)$</td>
<td>7) 6, 12, 18, 24 …</td>
</tr>
<tr>
<td>4) $a_n = 6 \left( \frac{1}{2} \right)^{n-1}$</td>
<td>D. $a_1 = 6, a_n = a_{n-1} + 3$</td>
<td>8) 6, -18, 54, -324 …</td>
</tr>
</tbody>
</table>

1) ________, ________
2) ________, ________
3) ________, ________
4) ________, ________
SEQUENCES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Given each of the following sequences defined by formulas, determine and label the first four terms. A variety of different notations is used below for practice purposes.

(a) \( f(n) = 7n + 2 \)  
(b) \( a_n = n^2 - 5 \)  
(c) \( t(n) = \left(\frac{2}{3}\right)^n \)  
(d) \( t_n = \frac{1}{n + 1} \)

2. Sequences below are defined recursively. Determine and label the next three terms of the sequence.

(a) \( f(1) = 4 \) and \( f(n) = f(n - 1) + 8 \)  
(b) \( a(n) = a(n - 1) \cdot \frac{1}{2} \) and \( a(1) = 24 \)

(c) \( b_n = b_{n-1} + 2n \) with \( b_1 = 5 \)  
(d) \( f(n) = 2f(n - 1) - n^2 \) and \( f(1) = 4 \)

4. A recursive sequence is defined by \( a_{n+1} = 2a_n - a_{n-1} \) with \( a_1 = 0 \) and \( a_2 = 1 \). Which of the following represents the value of \( a_5 \)?

(1) 8  
(2) -7  
(3) 3  
(4) 4

5. Which of the following formulas would represent the sequence 10, 20, 40, 80, 160, …

(1) \( a_n = 10^n \)  
(2) \( a_n = 10(2)^n \)  
(3) \( a_n = 5(2)^n \)  
(4) \( a_n = 2n + 10 \)
# Matching Sequences

**Directions:** Match the explicit formula, the recursive formula and the list of terms which all describe the SAME sequence.

<table>
<thead>
<tr>
<th>Explicit Formula</th>
<th>Recursive Formula</th>
<th>List of Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( a_n = 2n )</td>
<td>A ( a_1 = 3; \ a_n = 3 \cdot a_{n-1} )</td>
<td>11 ( 3, 6, 12, 24, ... )</td>
</tr>
<tr>
<td>2 ( a_n = 1 - 2n )</td>
<td>B ( a_1 = 2; \ a_n = 3 \cdot a_{n-1} )</td>
<td>12 ( 2, 4, 6, 8, ... )</td>
</tr>
<tr>
<td>3 ( a_n = 3^n )</td>
<td>C ( a_1 = 3; \ a_n = a_{n-1} + 5 )</td>
<td>13 ( 1, -6, 36, -216, ... )</td>
</tr>
<tr>
<td>4 ( a_n = 4n - 3 )</td>
<td>D ( a_1 = 0; \ a_n = a_{n-1} - 3 )</td>
<td>14 ( 2, 6, 18, 54, ... )</td>
</tr>
<tr>
<td>5 ( a_n = (-6)^{n-1} )</td>
<td>E ( a_1 = 3; \ a_n = 2 \cdot a_{n-1} )</td>
<td>15 ( -1, -3, -5, -7, ... )</td>
</tr>
<tr>
<td>6 ( a_n = 3n + 4 )</td>
<td>F ( a_1 = 2; \ a_n = a_{n-1} + 2 )</td>
<td>16 ( 1, 5, 9, 13, ... )</td>
</tr>
<tr>
<td>7 ( a_n = 3 \cdot 2^{n-1} )</td>
<td>G ( a_1 = 1; \ a_n = (-6) \cdot a_{n-1} )</td>
<td>17 ( 7, 10, 13, 16, ... )</td>
</tr>
<tr>
<td>8 ( a_n = 5n - 2 )</td>
<td>H ( a_1 = -1; \ a_n = a_{n-1} + 2 )</td>
<td>18 ( 3, 9, 27, 81, ... )</td>
</tr>
<tr>
<td>9 ( a_n = -3n + 3 )</td>
<td>I ( a_1 = 7; \ a_n = a_{n-1} + 3 )</td>
<td>19 ( 0, -3, -6, -9, ... )</td>
</tr>
<tr>
<td>10 ( a_n = 2 \cdot 3^{n-1} )</td>
<td>J ( a_1 = 1; \ a_n = a_{n-1} + 4 )</td>
<td>20 ( 3, 8, 13, 18, ... )</td>
</tr>
</tbody>
</table>

5. MATCHES:

1, _____, _____  5, _____, _____  9, _____, _____

2, _____, _____  6, _____, _____  10, _____, _____

3, _____, _____  7, _____, _____

4, _____, _____  8, _____, _____
APPLICATIONS

6. A tiling pattern is created from a single square and then expanded as shown. If the number of squares in each pattern defines a sequence, then determine the number of squares in the seventh pattern. Explain how you arrived at your choice. Can you write a recursive definition for the pattern?

![Pattern Image]

REASONING

7. Consider a sequence defined similarly to the Fibonacci, but with a slight twist:

\[ f(n) = f(n-1) - f(n-2) \]

with \( f(1) = 2 \) and \( f(2) = 5 \)

Generate terms \( f(3), f(4), f(5), f(6), f(7), \) and \( f(8) \). Then, determine the value of \( f(25) \).
LESSON #33 - ARITHMETIC AND GEOMETRIC SEQUENCES
COMMON CORE ALGEBRA II

In Common Core Algebra I, you studied two particular sequences known as arithmetic (based on constant addition to get the next term) and geometric (based on constant multiplying to get the next term). In this lesson, we will review the basics of these two sequences.

**ARITHMETIC SEQUENCE RECURSIVE DEFINITION**

Given \( f(1) \), then \( f(n) = f(n-1) + d \) or given \( a_1 \) then \( a_n = a_{n-1} + d \)

where \( d \) is called the **common difference** and can be positive or negative.

**ARITHMETIC SEQUENCE EXPLICIT FORMULA**

\[
 f(n) = f(1) + d(n-1) \quad \text{or} \quad a_n = a_1 + d(n-1)
\]

where \( d \) is called the **common difference** and can be positive or negative.

---

**Exercise #1:** Generate the next three terms of the given arithmetic sequence \( a_n = a_{n-1} + \frac{1}{2} \) and \( a_1 = \frac{3}{2} \).

**Exercise #2:** Find the first four terms of the given arithmetic sequence \( f(n) = 2 + 6(n-1) \).

**Exercise #3:** Consider \( f(n) = f(n-1) + 3 \) with \( f(1) = 5 \).

(a) Determine the value of \( f(2), f(3), \) and \( f(4) \).

(b) Write an explicit formula for the \( n^{\text{th}} \) term of an arithmetic sequence, \( f(n) \), based on the first term, \( f(1) \), \( d \) and \( n \).

(c) Using your answer to (b), find \( f(1), f(2), f(3), \) and \( f(4) \) to make sure you found the correct formula.
Exercise #4: Given that \( a_1 = 6 \) and \( a_4 = 18 \) are members of an arithmetic sequence,

(a) Find the value of \( d \), \( a_2 \), and \( a_3 \).

(b) Write a recursive definition for the \( n^{th} \) term of the sequence, \( a_n \).

(c) Write an explicit formula for the \( n^{th} \) term of an arithmetic sequence, \( a_n \).

(c) Determine the value of \( a_{20} \). Should you use your answer to (b) or (c) to do so? Explain.

Geometric sequences are defined very similarly to arithmetic, but with a multiplicative constant instead of an additive one.

<table>
<thead>
<tr>
<th>GEOMETRIC SEQUENCE RECURSIVE DEFINITION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given ( f(1) ) then ( f(n) = f(n-1) \cdot r ) or given ( a_1 ), then ( a_n = a_{n-1} \cdot r )</td>
</tr>
</tbody>
</table>

where \( r \) is called the common ratio and can be positive or negative and is often fractional.

<table>
<thead>
<tr>
<th>GEOMETRIC SEQUENCE EXPLICIT FORMULA</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(n) = f(1) \cdot r^{n-1} ) or ( a_n = a_1 \cdot r^{n-1} )</td>
</tr>
</tbody>
</table>

where \( r \) is called the common ratio and can be positive or negative and is often fractional.

Exercise #5: Generate the next three terms of the geometric sequence \( f(n) = f(n-1) \cdot \frac{1}{3} \) with \( f(1) = 9 \).
Exercise #6: Find the first four terms of the geometric sequence, \( t_n = (\sqrt{2})^n \). 

Exercise #7: Consider \( a_1 = 2 \) and \( a_n = a_{n-1} \cdot 3 \).

(a) Generate the value of \( a_4 \).

(b) Write an explicit formula for the \( n^{th} \) term of the sequence, \( a_n \), based on the first term, \( a_1 \), \( r \), and \( n \).

(c) Using your answer to (b), find \( a_1, a_2, a_3, \) and \( a_4 \) to make sure you found the correct formula.

Exercise #8: Given that \( f(1) = 6 \) and \( f(4) = 48 \) are members of a geometric sequence,

(a) Find the value of \( f(2) \) and \( f(3) \).

(b) Write a recursive definition for the \( n^{th} \) term of the sequence, \( f(n) \).

(c) Write an explicit formula for the \( n^{th} \) term of an arithmetic sequence, \( f(n) \).

(c) Determine the value of \( f(6) \). Should you use your answer to (b) or (c) to do so? Explain.
Exercise #9:

**PART 1: Directions:** Determine if each sequence is arithmetic or geometric, write an explicit formula for the sequence, and write a recursive formula for the sequence.

1. The terms of a sequence are $1, 3, 5, 7, 9, \ldots$

<table>
<thead>
<tr>
<th>Arithmetic or Geometric?</th>
<th>Explicit Formula</th>
<th>Recursive Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n =$</td>
<td>$a_1 =$ ; $a_n =$</td>
<td></td>
</tr>
</tbody>
</table>

2. The terms of a sequence are $2, 4, 8, 16, 32, \ldots$

<table>
<thead>
<tr>
<th>Arithmetic or Geometric?</th>
<th>Explicit Formula</th>
<th>Recursive Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n =$</td>
<td>$a_1 =$ ; $a_n =$</td>
<td></td>
</tr>
</tbody>
</table>

3. The terms of a sequence are $-2, 1, 4, 7, 10, \ldots$

<table>
<thead>
<tr>
<th>Arithmetic or Geometric?</th>
<th>Explicit Formula</th>
<th>Recursive Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n =$</td>
<td>$a_1 =$ ; $a_n =$</td>
<td></td>
</tr>
</tbody>
</table>

4. The terms of a sequence are $1, 10, 100, 1000, 10000, \ldots$

<table>
<thead>
<tr>
<th>Arithmetic or Geometric?</th>
<th>Explicit Formula</th>
<th>Recursive Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n =$</td>
<td>$a_1 =$ ; $a_n =$</td>
<td></td>
</tr>
</tbody>
</table>
**ARITHMETIC AND GEOMETRIC SEQUENCES**
**COMMON CORE ALGEBRA II HOMEWORK**

**FLUENCY**

Use the given information to fill in the other three rows in the table. **Hint:** If the terms are not given, find the first few terms before completing the rest of the row.

<table>
<thead>
<tr>
<th>Terms</th>
<th>Arithmetic or Geometric?</th>
<th>Explicit Formula</th>
<th>Recursive Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 10,14,18,22, . . .</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. 30,15,7.5,3.75, . . .</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td></td>
<td>$f(n) = 2 \times f(n-1)$</td>
<td>with $f(1) = 6$</td>
</tr>
<tr>
<td>4.</td>
<td></td>
<td>$a_n = a_{n-1} - 6$</td>
<td>with $a_1 = 20$</td>
</tr>
<tr>
<td>5.</td>
<td></td>
<td>$f(n) = 5 + \frac{1}{2}(n-1)$</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td></td>
<td>$a_n = 3(-4)^{n-1}$</td>
<td></td>
</tr>
</tbody>
</table>

7. Consider $f(n) = f(n-1) - 10$ with $f(1) = 24$.

(a) Determine the value of $f(2), f(3),$ and $f(4)$.

(b) Write an explicit formula for the $n^\text{th}$ term of the sequence, $f(n)$.

(c) Using your answer to (b), find $f(1), f(2), f(3),$ and $f(4)$ to make sure you found the correct formula.
8. Consider \( a_n = \frac{4}{3} \left( \frac{1}{3} \right)^{n-1} \).

(a) Determine the value of \( a_1, a_2, \text{ and } a_3 \).

(b) Write a recursive formula for the \( n^{th} \) term of the sequence.

(c) Using your answer to (b), find \( a_2 \) and \( a_3 \) to make sure you found the correct formula.

APPLICATIONS

10. The Koch Snowflake is a mathematical shape known as a fractal that has many fascinating properties. It is created by repeatedly forming equilateral triangles off of the sides of other equilateral triangles. Its first six iterations are shown to the right. The perimeters of each of the figures form a geometric sequence.

(a) If the perimeter of the first snowflake (the equilateral triangle) is 3, what is the perimeter of the second snowflake? Note: the dashed lines in the second snowflake are not to be counted towards the perimeter. They are only there to show how the snowflake was constructed.

(b) Given that the perimeters form a geometric sequence, what is the perimeter of the sixth snowflake? Express your answer to the nearest tenth.

(c) If the this process was allowed to continue forever, explain why the perimeter would become infinitely large.
LESSON #35 – SEQUENCE WORD PROBLEMS
COMMON CORE ALGEBRA II

Exercise #1: A store manager plans to offer discounts on some sweaters each week in an attempt to sell them before the winter season ends according to this sequence: $48, $36, $27, $20.25, . . . Write an explicit formula and a recursive formula for this sequence.

Exercise #2: After one customer buys 4 new tires, a garage recycling bin has 20 tires in it. After another customer buys four new tires, the bin has 24 tires in it. Write an explicit formula to represent the number of tires in the bin after \( n \) customers have bought four tires. How many tires would be in the bin after 9 customers buy four new tires?

Exercise #3: A pattern exists in the sum of the interior angles of polygons. The sum of the interior angles of a triangle is \( 180^\circ \), of a quadrilateral is \( 360^\circ \), and of a pentagon is \( 540^\circ \).

(a) Which choice is a recursive formula for this pattern?
   
   (1) \( f(1) = 180; f(n) = 180 \times f(n - 1) \)
   
   (2) \( f(1) = 180; f(n) = f(n+1) + 180 \)
   
   (3) \( f(1) = 180; f(n) = f(n - 1) + 360 \)
   
   (4) \( f(1) = 180; f(n) = f(n - 1) + 180 \)

(b) What is the sum of the interior angles of a nonagon (9 sides)?

Note: You can see that real world situations can be defined with recursive and explicit formulas. In certain situations it makes sense to include 0 in the domain of the function when there is a starting value at \( t=0 \).

Exercise #4: A certain culture of yeast increases by 50% every hour. There are 9 grams of yeast in a culture dish when the experiment begins. Write an explicit formula and recursive formula for the growth of the yeast \( h \) hours after the experiment begins.

Exercise #5: You have a cafeteria card worth $50. After you buy lunch on Monday its value is $46.75. After you buy lunch on Tuesday, its value is $43.50. Assuming the pattern continues, write a function rule to represent the amount of money left on the card. What is the value of the card after you buy 12 lunches?
**Exercise #6:** Lanie has decided to add strength training to her exercise program. Her trainer suggests that she add weight lifting for 5 minutes during her routine the first week. Each week thereafter, she is to increase the weight lifting time by 2 minutes. Which formula represents this sequential increase in weight lifting time?

(1) \( f(n) = 5n + 2 \)  
(2) \( f(n) = 3n + 2 \)  
(3) \( f(n) = 2n + 5 \)  
(4) \( f(n) = 2n + 3 \)

**Exercise #7:** The graph to the right models a sequence.

a. Does this function show a linear or exponential relationship?

b. Based on your answer to the previous question, find the common difference or common ratio.

c. Write an explicit formula for the sequence.

d. Write a recursive formula for the sequence.

**Exercise #8:** The graph to the right shows the number of teams left in the Women’s Basketball tournament at the beginning of each round.

a. Does this function show a linear or exponential relationship?

b. Based on your answer to the previous question, find the common difference or common ratio.

c. Write an explicit formula for the sequence.

d. Write a recursive formula for the sequence.
**Exercise #9**: Many of the exponential formulas we wrote last unit can be expressed as recursive formulas as well. \( t \) represents time in years. Round all answers to the nearest thousandth.

Note: A recursive formula will only calculate the amount after a whole number of years.

a. \( f(t) = 600e^{0.05t} \)
   
   i. Type of formula:

   ii. What is the value of \( f(0) \)?

   iii. Rewrite the formula in the form \( y = ab^t \)

   iv. Write the recursive definition for the formula. Why does it make more sense to start the definition with \( f(0) \)?

b. \( f(t) = 120 \left( 1 + \frac{0.06}{12} \right)^{12t} \)
   
   i. Type of formula:

   ii. Write a recursive definition for the function.

c. \( f(x) = 1200 \left( \frac{1}{2} \right)^{\frac{x}{14.5}} \)
   
   i. Type of formula:

   iii. Write a recursive definition for the function.
1. Suppose you are rehearsing for a concert. You plan to rehearse the piece you will perform four times the first day and then to double the number of times you rehearse the piece each day until the concert.
   a. Write a sequence of numbers to represent this situation.
   b. Is the sequence arithmetic, geometric, or neither?
   c. Write an explicit formula and a recursive formula for this sequence.
   d. If the concert was in 11 days, how many times would you rehearse the piece on the 10th day? Is this reasonable?

2. Write a recursive formula and an explicit formula for the sequence modeled in the graph to the right. (Assume the first point on the graph is (1,-3)).

3. A Greek theater has 30 seats in the first row of the center section. Each row behind the first row gains two additional seats. How many seats are in the 25th row? Write an explicit formula for this situation to find your answer.

4. A research lab is to begin experimentation with bacteria that grows by 20% each hour. The lab has 200 bacteria to begin the experiment.
   a. Write a recursive formula and an explicit formula to model the number of bacteria after n hours.
   b. How many whole bacteria will be present at the end of the 12th hour?
5. The summer Olympics occur every four years.
   a) Which formula represents the years of the summer Olympics, starting with 2016?

   \[ f(n) = 2016 + 4(n+1) \quad 2) f(n) = 4n + 2016 \]
   \[ 3) f(n) = 2016 + 4(n-1) \quad 4) f(n) = 2016 + (n+4) \]

6. Mr. Carlson suffers from allergies. When allergy season arrives, his doctor recommends that he take 300 mg of his medication the first day and decrease the dosage by one half each day for one week.
   a. Which rule represents his medication doses for the week?

   \[ f(n) = 300 \left( \frac{1}{2} \right)^{n-1} \quad (3) \quad f(n) = 300 + \frac{1}{2}n \]

   \[ (2) \quad f(n) = \left( \frac{1}{2} \right)^n \quad (4) \quad f(n) = \frac{1}{2}n \]

   b. To the nearest milligram, what is the amount of medication Mr. Carlson will take on the 7th day?

7. Your father wants you to help him build a shed in the backyard. He says he will pay you $50 for the first week and 5% more each week after that.
   a. Write an explicit formula and a recursive definition to model this situation.

   b. How much will your father pay you in the 5th week to the nearest cent?

8. Your grandmother gives you $1000 to start a college book fund. She tells you she will add $200 to the fund each month after that, if you will add $5 each month.

   a. Which "rule" generates a sequence of the monthly amounts in your college book fund?

   \[ f(n) = 1000 + 205(n-1) \quad 2) \quad f(n) = 205n + 795 \]

   \[ 3) f(1) = 1000; f(n) = f(n-1) + 205 \quad 4) \quad \text{All three formulas generate this sequence.} \]

   b. After how many months will the college book fund have $5715?
For each of the following formulas, state the type of formula and write a recursive definition for the formula. Round all answers to the nearest thousandth.

9. \( f(t) = 800 \left( 1 + \frac{0.04}{52} \right)^{52t} \)

10. \( f(t) = 400e^{-0.03t} \)

11. \( f(x) = 300 \left( \frac{1}{2} \right)^{\frac{t}{2.6}} \)
LESSON #36 - SUMMATION NOTATION  
COMMON CORE ALGEBRA II

Much of our work in this unit will concern adding the terms of a sequence. In order to specify this addition or summarize it, we introduce a new notation, known as summation or sigma notation that will represent these sums. This notation will also be used later in the course when we want to write formulas used in statistics.

**SUMMATION (SIGMA) NOTATION**

\[
\sum_{i=a}^{n} f(i) = f(a) + f(a+1) + f(a+2) + \ldots + f(n)
\]

where \( i \) is called the index variable, which starts at a value of \( a \), ends at a value of \( n \), and moves by unit increments (increase by 1 each time).

**Exercise #1:** Evaluate each of the following sums.

(a) \( \sum_{i=3}^{5} 2i^{-1} \)  
(b) \( \sum_{k=-1}^{3} k^2 \)  
(c) \( \sum_{j=1}^{5} (3(2^{j-1})) \)

(d) \( \sum_{i=1}^{5} (-1)^i \)  
(e) \( \sum_{k=1}^{4} (1+2(k-1)) \)  
(f) \( \sum_{i=1}^{3} i(i+1) \)

**Exercise #2:** Which of represents the value of \( \sum_{i=1}^{4} \frac{1}{i} \)?

(1) \( \frac{1}{10} \)  
(2) \( \frac{9}{4} \)  
(3) \( \frac{25}{12} \)  
(4) \( \frac{31}{24} \)
Exercise #3: Consider the sequence defined recursively by \( a_n = a_{n-1} + 2a_{n-2} \) and \( a_1 = 0 \) and \( a_2 = 1 \). Find the value of \( \sum_{i=4}^{7} a_i \).

Exercise #4: It is also good to be able to place sums into sigma notation. The values that are being summed in the next problems form either an arithmetic or geometric sequence. Look back at Exercise #1 on the previous page. Which problem represented the sum of the terms in an arithmetic sequence? A geometric sequence?

Exercise #5: Express each sum using sigma notation. Use \( n \) as your index variable. First determine if the sequence is arithmetic or geometric. Second, determine \( n \), the number of terms. Then use the appropriate explicit formula to write the sum.

(a) \( 40 + 28 + 16 + \ldots -20 \)

(b) \( \frac{1}{25} + \frac{1}{5} + 1 + 5 + \ldots 625 \)

(c) \( -6 + -3 + 0 + 3 + \ldots 15 \)

(d) \( -2 + 6 - 18 - 1458 \)

Exercise #6: Some sums are more interesting than others. Determine the value of \( \sum_{i=1}^{99} \left( \frac{1}{i} - \frac{1}{i+1} \right) \). Show your reasoning. This is known as a telescoping series (or sum).
SUMMATION NOTATION
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Evaluate each of the following. Place any non-integer answer in simplest rational form.

(a) \( \sum_{i=2}^{5} 4i \)

(b) \( \sum_{k=0}^{3} (k^2 + 1) \)

(c) \( \sum_{n=1}^{4} \frac{n}{n+1} \)

(d) \( \sum_{k=0}^{3} \frac{256}{1 - 2^k} \)

(e) \( \sum_{k=0}^{2} (-1)^{2k+1} \)

(f) \( \sum_{i=1}^{3} \log(10^i) \)

2. Which of the following is the value of \( \sum_{k=0}^{4} (4k+1) \)?

(1) 53  
(2) 45  
(3) 37  
(4) 80

3. The sum \( \sum_{i=4}^{7} 2^{i-7} \) is equal to

(1) \( \frac{15}{8} \)  
(2) \( \frac{3}{2} \)  
(3) \( \frac{3}{4} \)  
(4) \( \frac{7}{8} \)

4. Which of the following represents the sum \( 2 + 5 + 10 + \cdots + 82 + 101 \)?

(1) \( \sum_{j=1}^{6} (4j - 3) \)  
(2) \( \sum_{j=3}^{103} (j - 2) \)  
(3) \( \sum_{j=1}^{10} (j^2 + 1) \)  
(4) \( \sum_{j=0}^{11} (4^j + 1) \)
5. Express each sum using sigma notation. Use \( n \) as your index variable.

(a) \(-3 + 6 - 12 + 24 - 48 + \ldots \) 768

(b) \( \frac{1}{27} + \frac{1}{9} + \frac{1}{3} + \ldots \) 29

(c) 8.3 + 8.1 + 7.9 + 7.7 + \ldots \) for 20 terms

(d) 4 + 9 + 14 + \ldots 44 + 49

6. A sequence is defined recursively by the formula \( b_n = 4b_{n-1} - 2b_{n-2} \) with \( b_1 = 1 \) and \( b_2 = 3 \). What is the value of \( \sum_{i=3}^{5} b_i \)? Show the work that leads to your answer.

**REASONING**

7. A curious pattern occurs when we look at the behavior of the sum \( \sum_{k=1}^{n} (2k - 1) \).

(a) Find the value of this sum for a variety of values of \( n \) below:

\[ n = 2: \sum_{k=1}^{2} (2k - 1) = \]

\[ n = 4: \sum_{k=1}^{4} (2k - 1) = \]

\[ n = 3: \sum_{k=1}^{3} (2k - 1) = \]

\[ n = 5: \sum_{k=1}^{5} (2k - 1) = \]

(b) What types of numbers are you summing? What types of numbers are the sums?

(c) Find the value of \( n \) such that \( \sum_{k=1}^{n} (2k - 1) = 196 \).
LESSON #37 - GEOMETRIC SERIES
COMMON CORE ALGEBRA II

A series is simply the sum of the terms of a sequence. The fundamental definition/notion of a series is below.

**THE DEFINITION OF A SERIES**

If the set \{a_1, a_2, a_3, ...\} represent the elements of a sequence then the series, \( S_n \), is defined by:

\[
S_n = \sum_{i=1}^{n} a_i
\]

In truth, you have already worked extensively with series in previous lessons almost anytime you evaluated a summation problem.

**Exercise #1:** Given a geometric series defined by the recursive formula \( a_i = 3 \) and \( a_n = a_{n-1} \cdot 2 \), which of the following is the value of \( S_5 = \sum_{i=1}^{5} a_i \)?

- (1) 106
- (2) 75
- (3) 93
- (4) 35

**Exercise #2:** There is a formula for the sum of the first \( n \) terms of a geometric series, \( S_n \). The following is steps for deriving the formula.

<table>
<thead>
<tr>
<th>Steps</th>
<th>Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Write the explicit formula for a geometric sequence (( a_n ) form).</td>
<td>( a_n = )</td>
</tr>
<tr>
<td>2. Write out “n” terms of the sequence by plugging in 1 through “n”.</td>
<td></td>
</tr>
<tr>
<td>3. Write an equation, ( S_n ), which gives the sum of these terms.</td>
<td>( S_n = )</td>
</tr>
<tr>
<td>4. Multiply both sides of the equation by ( r ).</td>
<td></td>
</tr>
<tr>
<td>5. Find, in simplest form, the value of ( S_n - r \cdot S_n ) (step 3 minus step 4)</td>
<td>( S_n - r \cdot S_n = )</td>
</tr>
<tr>
<td>6. Write both sides of the equation in their factored form.</td>
<td></td>
</tr>
<tr>
<td>7. From the equation in step 6, find a formula for ( S_n ) in terms of ( a_1, r, ) and ( n ) by dividing by (1-r).</td>
<td></td>
</tr>
</tbody>
</table>
**Exercise #3:** Which of the following represents the sum of a geometric series with 8 terms whose first term is 3 and whose common ratio is 4?

1. 32,756  
2. 28,765  
3. 42,560  
4. 65,535

---

**SUM OF A FINITE GEOMETRIC SERIES**

For a geometric series defined by its first term, $a_1$, and its common ratio, $r$, the sum of $n$ terms is given by:

$$S_n = \frac{a_1(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a_1 - a_1r^n}{1-r}$$

**Exercise #4:** Find the value of the geometric series shown below. Show the calculations that lead to your final answer.

$$6 + 12 + 24 + \cdots + 768$$

---

**Exercise #5:** Maria places $500 at the beginning of each year into an account that earns 5% interest compounded annually. Maria would like to determine how much money is in her account after she has made her $500 deposit at the beginning of the 11th year (this amount would not get any interest).

(a) Determine a formula for the amount, $A(t)$, that a given $500 has grown to $t$-years after it was placed into this account.

(b) At the beginning of the 11th year, which will be worth more: the $500 invested in the first year or the fourth year? Explain by showing how much each is worth at the beginning of the 11th year.

(c) Based on (b), write a geometric sum representing the amount of money in Maria’s account after this time period. **Hint:** Write the terms starting with the last term – the term for the 11th year.

(d) Evaluate the sum in (c) using the formula above to the nearest cent.
**Exercise #6:** A person places 1 penny in a piggy bank on the first day of the month, 2 pennies on the second day, 4 pennies on the third, and so on. Will this person be a millionaire at the end of a 31 day month? Show the calculations that lead to your answer.
GEOMETRIC SERIES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Find the sums of geometric series with the following properties:
   
   (a) $a_1 = 6, r = 3$ and $n = 8$  
   
   (b) $a_1 = 20, r = \frac{1}{2}$, and $n = 6$  
   
   (c) $a_1 = -5, r = -2$, and $n = 10$

2. If the geometric series $54 + 36 + \cdots + \frac{128}{27}$ has seven terms in its sum then the value of the sum is

   (1) $\frac{4118}{27}$  
   
   (2) $\frac{1274}{3}$  
   
   (3) $\frac{1370}{9}$  
   
   (4) $\frac{8241}{54}$

3. A geometric series has a first term of 32 and a final term of $-\frac{1}{4}$ and a common ratio of $-\frac{1}{2}$. The value of this series is

   (1) 19.75  
   
   (2) 16.25  
   
   (3) 22.5  
   
   (4) 21.25

4. Which of the following represents the value of $\sum_{i=0}^{8} 256 \left(\frac{3}{2}\right)^i$? Think carefully about how many terms this series has in it.

   (1) 19,171  
   
   (2) 12,610  
   
   (3) 22,341  
   
   (4) 8,956

5. A geometric series whose first term is 3 and whose common ratio is 4 sums to 4095. The number of terms in this sum is

   (1) 8  
   
   (2) 5  
   
   (3) 6  
   
   (4) 4
6. Find the sum of the geometric series shown below. Show the work that leads to your answer.

\[
27 + 9 + 3 + \cdots \frac{1}{729}
\]

APPLICATIONS

7. In the picture shown at the right, the outer most square has an area of 16 square inches. All other squares are constructed by connecting the midpoints of the sides of the square it is inscribed within. Find the sum of the areas of all of the squares shown. First, consider the how the area of each square relates to the larger square that surrounds (circumscribes) it.

8. A college savings account is constructed so that $1000 is placed the account on January 1\textsuperscript{st} of each year with a guaranteed 3\% yearly return in interest, applied at the end of each year to the balance in the account. If this is repeatedly done, how much money is in the account after the $1000 is deposited at the beginning of the 19\textsuperscript{th} year to the nearest cent? Show the sum that leads to your answer as well as relevant calculations.

9. Jennifer finds a way to do her math homework more efficiently. The first night it takes her 32 minutes. Each night after that, it takes her \(\frac{3}{4}\) of the time it did the previous night. Was the total amount of time, to the nearest minute, she took doing her homework in the first seven nights? Write out the first five terms of this sum to help visualize.
LESSON #38 - MORTGAGE PAYMENTS  
COMMON CORE ALGEBRA II

Monthly mortgage payments can be found using the formula below. This formula comes from a geometric series, but we will just be learning how to work with the formula and solve for the different variables.

\[
M = \frac{P \left( \frac{r}{12} \right) \left( 1 + \frac{r}{12} \right)^n}{\left( 1 + \frac{r}{12} \right)^n - 1}
\]

*\( M \) = monthly payment  
*\( P \) = amount borrowed  
*\( r \) = annual interest rate  
*\( n \) = number of monthly payments

The most basic way to use the formula is to calculate monthly payments.

1. You took out a 30-year mortgage for $220,000 to buy a house. The interest rate on the mortgage is 5.2%. What are your monthly payments to the nearest cent?

When you are taking out a mortgage, you often know how much you can afford each month, and you want to determine what size mortgage you can afford.

2. Based on your current income, you can afford mortgage payments of $900 a month. You also want to take out a 15 year mortgage to pay off the loan sooner. If the average interest rate at this time is 3.375%, what size mortgage can you afford to the nearest cent?
The last way we will learn to use the mortgage payment formula involves determining the length of the loan that you can afford given the cost of a house, the amount you can spend per month, and the interest rate.

3. Imagine you have found the house of your dreams for $325,000. You know you can afford monthly mortgage payments of $1500. You qualified for a mortgage with an interest rate of 4.75%. Algebraically determine the number of payments you would need to make to pay off the loan at this rate to the nearest whole number. How many years would it take you to the nearest tenth of a year?

4. You are interested in purchasing a condo that costs $159,000. You know you can afford monthly mortgage payments of $900. You qualified for a mortgage with an interest rate of 3.825%. Algebraically determine the number of payments you would need to make to pay off the loan at this rate to the nearest whole number. How many years would it take you to the nearest tenth of a year?
5. You took out a 20-year mortgage for $180,000 to buy a house. The interest rate on the mortgage is 4.8%.
   a. What are your monthly payments to the \textit{nearest dollar}?

   b. With this monthly payment, what is the total cost to pay off the loan?

6. Based on your current income, you can afford mortgage payments of $1100 a month. You also want to take out a 30-year mortgage to spread the payments out over time. If the average interest rate at this time is 5.25%, what size mortgage can you afford to the \textit{nearest dollar}?

7. You have chosen a starter home for $120,000. You know you can afford monthly mortgage payments of $700. You qualified for a mortgage with an interest rate of 3.125%. Algebraically determine the number of payments you would need to make to pay off the loan at this rate to the \textit{nearest whole number}. How many years would it take you to the \textit{nearest tenth of a year}?
MORTGAGE PAYMENTS
COMMON CORE ALGEBRA II HOMEWORK

1. You took out a 15-year mortgage for $160,000 to buy a house. The interest rate on the mortgage is 5.2%.
   a. What are your monthly payments to the nearest cent?

2. Based on your current income, you can afford mortgage payments of $1250 a month. You also want to take out a 25-year mortgage to spread the payments out over time. If the average interest rate at this time is 3.375%, what size mortgage can you afford to the nearest dollar?

3. You have chosen a home in the perfect location for $250,000. You know you can afford monthly mortgage payments of $1,400. You qualified for a mortgage with an interest rate of 4.75%. Algebraically determine the number of payments, to the nearest whole payment, you would need to make to pay off the loan at this rate. How many years would it take you to the nearest tenth of a year?
4. You took out a 30-year mortgage for $350,000 to buy a house. The interest rate on the mortgage is 4.5%.
   a. What are your monthly payments to the nearest dollar?

b. With this monthly payment, what is the total cost to pay off the loan?

5. You found a small home for $130,000. You know you can afford monthly mortgage payments of $990. You qualified for a mortgage with an interest rate of 4.0%. Algebraically determine the number of payments, to the nearest whole payment, you would need to make to pay off the loan at this rate. How many years would it take you to the nearest tenth of a year?